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ε-GENERAL ORTHOGONALITY AND ε-ORTHOGONALITY IN NORMED LINEAR SPACES

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Abstract

In this paper, we investigate some results on ε-general orthogonality in normed spaces similar to ε-orthogonality. In this paper we shall show that ε-general orthogonality and ε-orthogonality has coincided, and with this new results, we obtain some new results on ε-best approximation.

1. Introduction

Let B be a normed space. In a normed linear space, we used to consider ϵ-orthogonality. The purpose of this paper is to obtain a new

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concept of ε-orthogonality and this concept have very application in approximation theory (see [1-7]). In [5], we have for a normed space B, if $x, y \in B$ and $\varepsilon > 0$, x is said to be ε-orthogonal to y and is denoted by $x \perp_{\varepsilon}^{BG} y$ if and only if $\| x \| \leq \|x + \alpha y\| + \varepsilon$ for any scaler α. Also, if G is a linear subspace of a normed linear space X, $x \in X \setminus \overline{G}$ and $g_0 \in G$. Then g_0 is a ε-best approximation of x if and only if $\|x - g_0\| \leq dist(x, G) + \varepsilon$ or equivalence $x - g_0 \perp_{\varepsilon}^{BG} g$ for all $g \in G$.

At first we state the following lemma which is needed in the proof of the main results.

Lemma 1.1 ([6]). *Let B be a normed linear space, G is a linear subspace of B, $x \in B \setminus \overline{G}$ and $g_0 \in G$ and $\varepsilon > 0$. Then the following two conditions are equivalent*:

(1) *g_0 is a ε-best approximation of x.*

(2) *There exists a linear functional Λ on X such that, $\|\Lambda\| = 1$, $\Lambda(x - g_0) \leq dist(x, G) + \varepsilon$ and $\Lambda(y) = 0$ for all $y \in G$.*

In this note, we shall consider ε-general orthogonality in the Banach spaces.

Definition 1.2. Let B be a real Banach space, $x, y \in B$ and $\varepsilon > 0$. Then, we say that $x \perp_{\varepsilon}^{G} y$ if there exists a unique $\phi_x \in B^*$ such that $\phi_x(x) \geq \| x \|^2 - \varepsilon \| x \|$, $\|\phi_x\| = \| x \|$ and $\phi(y) = 0$.

2. Main Results

In this section we state and prove some characterizations of the ε-general orthogonality and ε-orthogonality in normed spaces.

Lemma 2.1. *Let B be a normed linear space, $x, y \in X$ and $\varepsilon > 0$. Then the following statements are true*:

1) $x \perp_{\varepsilon}^{BG} y$.

2) *there exists* $\Lambda \in B^*$ *such that* $\|\Lambda\| = 1,\ \Lambda(x) \geq \| x \| - \varepsilon$ *and* $\Lambda(y) = 0.$

Proof. If $x \perp_{\epsilon}^{BG} y$ then 0 is a ε-best approximation of x in subspace $< y >$. From Lemma 1.1, we have there exists a linear functional Λ on X such that, $\|\Lambda\| = 1,\ \Lambda(x) \leq dist(x, G) + \varepsilon$ and $\Lambda(y) = 0.$

If there exists a linear functional Λ on X such that, $\|\Lambda\| = 1,\ \Lambda(x) \leq dist(x, G) + \varepsilon$ and $\Lambda(y) = 0.$ Then

$$\| x \| \leq \Lambda(x) + \epsilon \leq \|\Lambda\|\|x + \alpha y\| + \varepsilon = \|x + \alpha y\| + \varepsilon.$$ □

In the following theorem, we show that ε-general orthogonality and ε-orthogonality has coincided.

Theorem 2.2. *Let B be a normed linear space,* $x, y \in X$ *and* $\varepsilon > 0.$ *Then the following statements are true*:

1) $x \perp_{\varepsilon}^{BG} y.$

2) $x \perp_{\varepsilon}^{G} y.$

Proof. Suppose $x \perp_{\varepsilon}^{BG} y,$ from Lemma 2.1, then there exists $\Lambda \in B^*$ such that $\|\Lambda\| = 1,\ \Lambda(x) \geq \| x \| - \varepsilon$ and $\Lambda(y) = 0.$ Consider $\Lambda' = \| x \|\Lambda,$ then Λ' is a linear functional on $B, \|\Lambda'\| = \| x \|,\ \Lambda'(x) \geq \| x \|^2 -\varepsilon \| x \|$ and $\Lambda'(y) = 0.$ It follows that $x \perp_{\varepsilon}^{G} y.$

If $x \perp_{\varepsilon}^{G} y,$ then there exists $\Lambda \in B^*$ such that $\|\Lambda\| = \| x \|,\ \Lambda(x) \geq \| x \|^2 - \varepsilon \| x \|$ and $\Lambda(y) = 0.$ Consider $\Lambda' = \frac{\Lambda}{\| x \|},$ then Λ' is a linear functional on $B, \|\Lambda'\| = 1,\ \Lambda'(x) \geq \| x \| - \varepsilon$ and $\Lambda'(y) = 0.$ From Lemma 2.1, It follows that $x \perp_{\varepsilon}^{BG} y.$

Theorem 2.3. *Let B be a normed linear space and* $\varepsilon > 0.$ *Then the following statements are true*:

1) *For all* $x, y \in X$ *and* $x \perp_{\varepsilon}^{G} y$, *then* $x \perp_{\delta}^{G} y$ *for every* $\delta \geq \varepsilon$.

2) *For all* $x \in B$ *and* $0 < \varepsilon < \| x \|$ *if* $x \perp_{\varepsilon}^{G} x$, *then* $x = 0$.

3) *For all* $x, y \in B$ *and* $0 < \varepsilon < \| x \|$ *if* $x \perp_{\varepsilon}^{G} y$ *and* $x \neq 0$, *then*

$$< x > \bigcap < y > = \{0\}.$$

4) *For all* $x \in X$, $0 \perp_{\varepsilon}^{G} x$ *and* $x \perp_{\varepsilon}^{G} 0$.

5) *For all* $x, y \in X$, *if for every* $\varepsilon > 0$, $x \perp_{\varepsilon}^{G} y$, *then* $\| x \| \leq \|x + \alpha y\|$.

Proof. (1). From Theorem 2.1 since $x \perp_{\varepsilon}^{G} y$ we have $x \perp_{\varepsilon}^{BG} y$, therefore $\| x \| \leq \|x + \alpha y\| + \varepsilon$. If $\delta \geq \varepsilon$, then $\| x \| \leq \|x + \alpha y\| + \delta$, that is, $x \perp_{\delta}^{BG} y$ and $x \perp_{\delta}^{G} y$.

(2). For all $x \in B$, if $x \perp^{G} x$. Then $\phi_x(x) = 0$ and $\phi_x(x) \geq \| x \|^2 - \varepsilon \| x \|$, because $0 < \varepsilon < \| x \|$, we have $x = 0$.

(3). If $z \in < x > \bigcap < y >$, then for scales c_1, c_2, $z = c_1 x = c_2 y$. Hence, $\phi_x(z) = 0$, it follows that $\phi_x(c_1 x) = 0$. If $c_1 \neq 0$, it follows that $\phi_x(x) = 0$,

because $0 < \varepsilon < \| x \|$ we have $x = 0$ and a counteraction. Then $c_1 = 0$ and $z = 0$.

(4). Because $0 = \|0\| = \|0 + \alpha x\| + \varepsilon$ therefore $0 \perp_{\varepsilon}^{BG} x$ and $0 \perp_{\varepsilon}^{G} x$. Also $\| x \| \leq \|x + \alpha 0\| + \varepsilon$ therefore $x \perp_{\varepsilon}^{BG} 0$ and $x \perp_{\varepsilon}^{G} 0$.

(5). It is clear. □

Theorem 2.4. *Let B be a normed linear space, G a linear subspace of* B, $x \in B \setminus \overline{G}$ *and* $M \subseteq G$ *and* $\varepsilon > 0$. *Then the following two conditions are equivalent*:

1) $M \subseteq \{g_0 \in G : \|x - g_0\| = dist(x, G)\}$.

2) *There exists a* $\phi_x \in B^*$ *such that for every* $g_0 \in M$, *we have*

$$\phi_x(x - g_0) \geq \|x - g_0\|^2 - \varepsilon\|x - g_0\| \text{ and } \|\phi_x\| = \|x - g_0\|,$$

and $\phi(y) = 0$ *for every* $y \in G$.

Proof. $1) \Rightarrow 2)$ If $M \subseteq \{g_0 \in G : \|x - g_0\| = dist(x, G)\}$, then

$$M \subseteq \{g_0 \in G : \|x - g_0\| \leq dist(x, G) + \varepsilon\}.$$

Therefore for $g \in M$, we have $\|x - g\| \leq dist(x, G) + \varepsilon$, from Lemma 1.1, it follows that there exists a linear functional Λ on X such that, $\|\Lambda\| = 1$, $\Lambda(x - g) \leq dist(x, G) + \varepsilon$ and $\Lambda(y) = 0$ for all $y \in G$. If $g' \in M$ be a another element, from Lemma 1.1, it follows that $\Lambda(x - g') = \Lambda(x - g) \leq dist(x, G) + \varepsilon$ and $\|\Lambda\| = \|x - g'\| = \|x - g\|$.

$2) \Rightarrow 1)$. It is trivial.

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